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Research article

Introduction for SuperHyperGraph Labeling and MultiLabeling

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Abstract

A finite *hypergraph* generalizes the classical graph model by allowing *hyperedges* that can connect any nonempty subset of vertices. Building on this foundation, a finite *SuperHyperGraph* is obtained through iterative application of the powerset construction, thereby creating nested families of vertex and edge sets that capture multi-layered relationships. Graph labeling assigns numbers or symbols to vertices and/or edges of a graph under rules, modeling constraints, optimization, or communication. In this paper, we define and study the mathematical properties of *Graph Labeling*, *HyperGraph Labeling*, *SuperHyperGraph Labeling*, *Graph MultiLabeling*, *HyperGraph MultiLabeling*, and *SuperHyperGraph MultiLabeling*.

1. Preliminaries

This section fixes the terminology and notation used throughout the paper. Unless stated otherwise, every graph considered here is *finite*.

1.1. Labeling Graph

Graph theory investigates mathematical models built from vertices and edges that capture pairwise relations, network structure, and connectivity [1, 2]. Graph labeling assigns numbers or symbols to vertices and/or edges of a graph under rules, modeling constraints, optimization, or communication [3–6].

Definition 1.1 (Graph labeling)**.** *[3–6] Let $G = (V, E)$ be a finite (simple) graph. A* graph labeling *is a choice of label sets L_V (for vertices) and L_E (for edges), together with functions*

$$\ell_V : V \to L_V \quad \text{and/or} \quad \ell_E : E \to L_E.$$

A labeling may be required to satisfy additional constraints, depending on the context. Typical instances include:

- Proper vertex-coloring: $L_V = \{1, \ldots, k\}$ *and for every edge* $uv \in E$, $\ell_V(u) \neq \ell_V(v)$.
- $L(p,q)$-labeling: $L_V \subseteq \mathbb{Z}$ *such that for adjacent* u, v *one has* $|\ell_V(u) - \ell_V(v)| \geq p$*, and for vertices at distance* 2 *one has* $|\ell_V(u) - \ell_V(v)| \geq q$.

- Edge labelings *(e.g., graceful, harmonious): constraints are imposed on* ℓ_E *(and sometimes on* ℓ_V*) to control the multiset of induced values.*

When only one of ℓ_V *or* ℓ_E *is present, we speak of a* vertex labeling *or an* edge labeling, *respectively.*

Example 1.2 ($L(2,1)$-labeling of a path)**.** *Consider the path* P_5 *with vertices* v_1, v_2, v_3, v_4, v_5 *and edges* $v_i v_{i+1}$ $(i = 1, \dots, 4)$*. Define a vertex labeling* $\ell_V : V(P_5) \to \mathbb{Z}$ *by*

$$\ell_V(v_1) = 0, \quad \ell_V(v_2) = 2, \quad \ell_V(v_3) = 4, \quad \ell_V(v_4) = 1, \quad \ell_V(v_5) = 3.$$

We verify the $L(2,1)$ *constraints:*

- *For each edge* $v_i v_{i+1}$*, the label gap is at least* 2*:*

$$|\ell_V(v_1) - \ell_V(v_2)| = |0-2| = 2, \; |\ell_V(v_2) - \ell_V(v_3)| = |2-4| = 2,$$
$$|\ell_V(v_3) - \ell_V(v_4)| = |4-1| = 3, \; |\ell_V(v_4) - \ell_V(v_5)| = |1-3| = 2.$$

- *For distance-2 pairs, the gap is at least* 1*:*

$$|\ell_V(v_1) - \ell_V(v_3)| = |0-4| = 4,$$
$$|\ell_V(v_2) - \ell_V(v_4)| = |2-1| = 1,$$
$$|\ell_V(v_3) - \ell_V(v_5)| = |4-3| = 1.$$

Hence ℓ_V *is a valid* $L(2,1)$*-labeling of* P_5*.*

1.2. SuperHyperGraphs

A finite *hypergraph* extends the classical notion by allowing *hyperedges* to join arbitrary nonempty subsets of the vertex set, thereby representing multiway interactions [7–10]. Pushing this idea further, a finite *SuperHyperGraph* arises by iterating the powerset construction, which yields nested families of vertex- and edge-sets and thus encodes multi-layer relationships [11–16]. Such models are useful in, for example, molecular design, complex-network analysis, and more applications [17–24]. Unless stated otherwise, the index n in $\mathcal{P}_n(\cdot)$ and in an n–SuperHyperGraph is taken to be nonnegative.

Definition 1.3 (Base set)**.** *A* base set *S is the ambient universe of discourse:*

$$S = \{ x \mid x \text{ belongs to the context under consideration} \}.$$

Every object occurring in $\mathcal{P}(S)$ *or in any iterated powerset* $\mathcal{P}_n(S)$ *is, by definition, a subset ultimately formed from elements of* S*.*

Definition 1.4 (Powerset)**.** *(see [25–27]) For a set* S*, the* powerset $\mathcal{P}(S)$ *is the collection of all subsets of* S*:*

$$\mathcal{P}(S) = \{ A \subseteq S \}.$$

In particular, both the empty set $\emptyset$ *and* S *itself lie in* $\mathcal{P}(S)$*.*

Definition 1.5 (Hypergraph)**.** *[28, 29] A* hypergraph *is an ordered pair* $H = (V,E)$ *with*

- *a finite vertex set* V*, and*
- *a finite family* E *of nonempty subsets of* V*, whose members are called* hyperedges*.*

Hypergraphs naturally encode interactions involving more than two participants.

Example 1.6 (Hypergraph — project teams sharing resources (real life))**.** *Let the employees be the vertex set*

$$V = \{\text{Alice}, \text{Bob}, \text{Chen}, \text{Dina}\}.$$

Define the family of hyperedges

$$E = \big\{ \{\text{Alice}, \text{Bob}, \text{Chen}\}, \; \{\text{Bob}, \text{Dina}\}, \; \{\text{Alice}, \text{Dina}\} \big\}.$$

Interpretation. Each hyperedge is a team *that jointly uses a shared resource (e.g., a meeting room or a code repository). Thus* $H = (V,E)$ *is a finite hypergraph: it records not only pairwise collaborations (*$\{\text{Bob}, \text{Dina}\}$*) but also a three-person collaboration (*$\{\text{Alice}, \text{Bob}, \text{Chen}\}$*).*

Definition 1.7 (n-th powerset)**.** *[30–34] For a set* X*, define* $\mathcal{P}_1(X) = \mathcal{P}(X)$ *and, for* $n \geq 1$*,*

$$\mathcal{P}_{n+1}(X) = \mathcal{P}\big(\mathcal{P}_n(X)\big).$$

When excluding the empty set, write $\mathcal{P}_n^*(X) = \mathcal{P}_n(X) \setminus \{\emptyset\}$*.*

Example 1.8 (n-th powerset — explicit small instance)**.** *Take* $X = \{p,q\}$*. Then*

$$\mathcal{P}_1(X) = \mathcal{P}(X) = \{\varnothing, \{p\}, \{q\}, \{p,q\}\}.$$

The second-level powerset is the powerset of this 4*-element set, hence* $|\mathcal{P}_2(X)| = 2^4 = 16$*, for example it contains*

$$\{\{p\}\}, \quad \{\{q\}\}, \quad \{\{p\},\{q\}\}, \quad \{\varnothing, \{p\}, \{q\}, \{p,q\}\} = \mathcal{P}_1(X).$$

If we exclude the empty set at each step, then

$$\mathcal{P}_1^*(X) = \{\{p\},\{q\},\{p,q\}\}, \qquad \mathcal{P}_2^*(X) = \mathcal{P}_2(X) \setminus \{\varnothing\}.$$

This illustrates how iterating $\mathcal{P}(\cdot)$ *builds higher "layers" of set families.*

Definition 1.9 (n–SuperHyperGraph)**.** *(see [35, 36]) Let V_0 be a finite, nonempty base set and define*

$$\mathscr{P}^0(V_0)V_0, \qquad \mathscr{P}^{k+1}(V_0)\mathscr{P}\big(\mathscr{P}^k(V_0)\big) \quad (k \in \mathbb{N}).$$

*For $n \geq 0$, an n–*SuperHyperGraph on V_0 *is a pair*

$$\mathrm{SHG}^{(n)} = (V, E)$$

with

$$V \subseteq \mathscr{P}^n(V_0) \quad \textit{and} \quad E \subseteq \mathscr{P}(V) \setminus \{\emptyset\}.$$

*Members of V are the n–*supervertices*, while members of E are the n–*superedges *(each n–superedge is a nonempty subset of V).*

Example 1.10 (n–SuperHyperGraph — families of task-sets (real life))**.** *Let the base set of atomic tasks be $V_0 = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$. Then $\mathscr{P}^1(V_0) = \mathscr{P}(V_0)$ consists of all task-sets, and $\mathscr{P}^2(V_0) = \mathscr{P}(\mathscr{P}(V_0))$ consists of* families *of task-sets. Choose the $n = 2$ supervertex set*

$$V = \Big\{ F_1,\, F_2 \Big\}, \qquad F_1 = \big\{\{\mathrm{a}\}, \{\mathrm{b}\}\big\}, \quad F_2 = \big\{\{\mathrm{b}\}, \{\mathrm{c}\}\big\}.$$

Define the superedge family

$$E = \big\{\ \{F_1\},\ \{F_2\},\ \{F_1, F_2\}\ \big\} \ \subseteq\ \mathscr{P}(V) \setminus \{\varnothing\}.$$

Interpretation. Each supervertex F_i is a plan family *(a set of admissible task-sets); a superedge groups one or several plan families that are considered together (e.g., combined scenarios). Hence* $\mathrm{SHG}^{(2)} = (V, E)$ *is a finite 2–SuperHyperGraph on V_0.*

2. Main Results

This section presents the principal findings of the paper.

2.1. Hypergraph labeling

HyperGraph labeling assigns labels to vertices and hyperedges of hypergraphs, encoding multi-participant interactions, scheduling, resource distribution, or network optimization (cf.[37–41]).

Definition 2.1 (Primal (2-section) graph)**.** *Given a hypergraph $H = (V, E)$, its* primal graph *(also called the* 2-section*) is*

$$G(H) := \big(V,\, E'\big), \qquad E' := \big\{\ \{u, v\} \subseteq V\ \big|\ \exists e \in E \text{ with } \{u, v\} \subseteq e\ \big\}.$$

Distances between vertices of H are measured in $G(H)$ and denoted dist_H.

Definition 2.2 (Hypergraph labeling (schema-based))**.** *Let $H = (V, E)$ be a hypergraph, and let L_V, L_E be nonempty label sets (for vertices and hyperedges). A* (vertex/edge) labeling *of H is a pair of maps*

$$\ell_V : V \to L_V, \qquad \ell_E : E \to L_E,$$

where either map may be omitted if not used. A hypergraph labeling schema *is a first-order predicate $\Phi(H, \ell_V, \ell_E)$ built from the incidence relation "$v \in e$", the distance dist_H on V (Definition 2.1), the equality/inequality on labels, and quantification over V and E. We say that (ℓ_V, ℓ_E) is a* valid hypergraph labeling (for Φ) *if $\Phi(H, \ell_V, \ell_E)$ holds.*

Remark 2.3 (Classical graph labelings as instances of Φ)**.** *Typical choices of Φ recover familiar graph labelings when H is 2-uniform:*

- ***Proper vertex coloring:*** *$L_V = \{1, \ldots, k\}$ and $\Phi \equiv (\forall \{u, v\} \in E)\ \ell_V(u) \neq \ell_V(v)$.*
- *$L(p, q)$**-labeling:** $L_V \subseteq \mathbb{Z}$ and*

$$\Phi \equiv \big(\forall u \neq v \in V\big) \Big(\mathrm{dist}_H(u, v) = 1 \Rightarrow |\ell_V(u) - \ell_V(v)| \geq p\ \wedge\ \mathrm{dist}_H(u, v) = 2 \Rightarrow |\ell_V(u) - \ell_V(v)| \geq q\Big).$$

- ***Strong hypergraph coloring (a genuine hypergraph constraint):*** *$L_V = \{1, \ldots, k\}$ and $\Phi \equiv (\forall e \in E)$ the labels $\{\ell_V(v) \mid v \in e\}$ are pairwise distinct.*

Example 2.4 (Strong hypergraph 3-coloring of a small hypergraph)**.** *Let $H = (V, E)$ with*

$$V = \{1, 2, 3, 4\}, \qquad E = \big\{\{1, 2, 3\},\ \{3, 4\}\big\}.$$

Take $L_V = \{r, g, b\}$ and let Φ be "strong coloring": for every $e \in E$ the labels on e are pairwise distinct. Define

$$\ell_V(1) = r, \quad \ell_V(2) = g, \quad \ell_V(3) = b, \quad \ell_V(4) = r.$$

Verification. *On $e_1 = \{1, 2, 3\}$ we have $\{r, g, b\}$ (all distinct). On $e_2 = \{3, 4\}$ we have $\{b, r\}$ (distinct). Thus ℓ_V satisfies Φ and is a valid strong 3-coloring of H.*

Example 2.5 ($L(2,1)$-labeling lifted to a hypergraph via the 2-section)**.** *Let $H = (V,E)$ with*

$$V = \{a,b,c,d,e\}, \qquad E = \big\{\{a,b,c\},\ \{c,d\},\ \{d,e\}\big\}.$$

Its 2-section $G(H)$ has edges

$$\{a,b\},\ \{a,c\},\ \{b,c\},\ \{c,d\},\ \{d,e\}.$$

Let $L_V = \mathbb{Z}$ and let Φ be the hypergraph version of $L(2,1)$: for all distinct $u,v \in V$,

$$\mathrm{dist}_H(u,v) = 1 \ \Rightarrow\ |\ell_V(u) - \ell_V(v)| \geq 2, \qquad \mathrm{dist}_H(u,v) = 2 \ \Rightarrow\ |\ell_V(u) - \ell_V(v)| \geq 1,$$

where dist_H is computed in $G(H)$. Define labels

$$\ell_V(a) = 0,\ \ \ell_V(b) = 2,\ \ \ell_V(c) = 4,\ \ \ell_V(d) = 1,\ \ \ell_V(e) = 3.$$

Adjacency checks *(distance 1 in $G(H)$):*

$$|0-2| = 2\ (\{a,b\}),\ \ |0-4| = 4\ (\{a,c\}),\ \ |2-4| = 2\ (\{b,c\}),\ \ |4-1| = 3\ (\{c,d\}),\ \ |1-3| = 2\ (\{d,e\}).$$

Distance-2 checks *(one example per class):*

$$a\text{-}d:\ a{-}c{-}d \Rightarrow |0-1| = 1; \quad b\text{-}d:\ b{-}c{-}d \Rightarrow |2-1| = 1;$$

$$a\text{-}e:\ a{-}c{-}d{-}e \Rightarrow |0-3| = 3; \quad b\text{-}e:\ b{-}c{-}d{-}e \Rightarrow |2-3| = 1.$$

All constraints hold, hence ℓ_V is a valid $L(2,1)$-labeling of the hypergraph H under Φ. When H is 2-uniform, this reduces to the classical $L(2,1)$-labeling of a graph.

Theorem 2.6 (Hypergraph labeling strictly generalizes graph labeling)**.** *Let (G,Ψ) be any graph-labeling problem on a simple graph $G = (V,E_G)$, where $\Psi(G,\lambda)$ is a predicate expressed using adjacency/distance in G and (in)equalities among labels. Regard G as the 2-uniform hypergraph $H = (V,E)$ with $E := \{\{u,v\} \mid uv \in E_G\}$. Define the hypergraph labeling schema $\Phi(H,\ell_V) := \Psi\big(G(H),\ell_V\big)$. Then the following are equivalent:*

$$\exists \lambda : V \to L \text{ with } \Psi(G,\lambda) \iff \exists \ell_V : V \to L \text{ with } \Phi(H,\ell_V).$$

Consequently, every graph labeling instance is an instance of hypergraph labeling; moreover, hypergraph-only constraints (e.g. strong hypergraph coloring) have no counterpart on graphs with only size-2 edges, so the inclusion is strict.

Proof. ($\Rightarrow$) Suppose $\lambda : V \to L$ satisfies $\Psi(G,\lambda)$. Since $G(H) = G$ by construction of E from E_G, setting $\ell_V := \lambda$ yields

$$\Phi(H,\ell_V) \ \equiv\ \Psi\big(G(H),\ell_V\big) \ =\ \Psi(G,\lambda),$$

so ℓ_V is a valid hypergraph labeling.
($\Leftarrow$) Conversely, if $\ell_V : V \to L$ satisfies $\Phi(H,\ell_V)$, then $\Psi\big(G(H),\ell_V\big)$ holds. But $G(H) = G$, hence $\Psi(G,\ell_V)$ holds, and $\lambda := \ell_V$ is a valid graph labeling.
Strictness follows because there exist valid Φ that quantify over hyperedges of size ≥ 3 (e.g. strong hypergraph coloring in Remark 2.3), which impose constraints on $|e|$-tuples of vertices inside a single hyperedge; such constraints cannot be expressed on a simple graph whose edges are only size-2 subsets without passing to cliques or auxiliary gadgets. □

2.2. SuperHyperGraph Labeling

SuperHyperGraph labeling assigns structured labels to vertices and superedges across nested powerset levels, capturing hierarchical multi-layered relationships and advanced constraints.

Definition 2.7 (SuperHyperGraph labeling (schema–based))**.** *Let $\mathsf{SHG}^{(n)} = (V,E)$ and let L_V, L_E be nonempty label sets for vertices and superedges. A* (vertex/edge) labeling *is a pair of maps*

$$\ell_V : V \to L_V, \qquad \ell_E : E \to L_E,$$

where either map may be omitted if not needed. A labeling schema *is a first–order predicate*

$$\Phi\big(\mathsf{SHG}^{(n)},\ell_V,\ell_E\big)$$

built from the incidence relation "$v \in e$" ($v \in V,\ e \in E$), equality/inequality on labels, the distance $\mathrm{dist}_{\mathsf{SHG}}$ from the Definition, and (optionally) structural predicates on n–level objects (e.g. cardinalities, inclusion between members of $V \subseteq \mathscr{P}^n(V_0)$). We call (ℓ_V,ℓ_E) a valid SuperHyperGraph labeling *(for Φ) if $\Phi(\mathsf{SHG}^{(n)},\ell_V,\ell_E)$ holds.*

Remark 2.8 (Recovering classical schemas)**.** *By choosing Φ appropriately one recovers many standard labeling families:*

- ***Proper vertex coloring:*** *$L_V = \{1,\dots,k\}$ and $(\forall\{u,v\} \in E')\ \ell_V(u) \neq \ell_V(v)$, where E' is from the 2–section.*
- *$L(p,q)$**–labeling:** $L_V \subseteq \mathbb{Z}$ and, for all distinct $u,v \in V$,*

$$\mathrm{dist}_{\mathsf{SHG}}(u,v) = 1 \Rightarrow |\ell_V(u) - \ell_V(v)| \geq p, \quad \mathrm{dist}_{\mathsf{SHG}}(u,v) = 2 \Rightarrow |\ell_V(u) - \ell_V(v)| \geq q.$$

- ***Strong hypercoloring (genuinely hyper):*** *$L_V = \{1,\dots,k\}$ and $(\forall e \in E)$ the labels $\{\ell_V(v) : v \in e\}$ are pairwise distinct.*

Example 2.9 (Classical $L(2,1)$ as a SuperHyperGraph labeling (the case $n=0$)). *Let $G=P_5$ be the path on vertices $v_1,\dots,v_5$. Form the 0–SuperHyperGraph $\mathsf{SHG}^{(0)}=(V,E)$ with $V=\{v_1,\dots,v_5\}$ and $E=\big\{\{v_i,v_{i+1}\}:i=1,2,3,4\big\}$. Let Φ be the $L(2,1)$ schema of Remark 2.8 with $L_V=\mathbb{Z}$ and distances taken in $G(\mathsf{SHG}^{(0)})=G$. Define*

$$\ell_V(v_1)=0,\quad \ell_V(v_2)=2,\quad \ell_V(v_3)=4,\quad \ell_V(v_4)=1,\quad \ell_V(v_5)=3.$$

Adjacency (distance 1) checks: $|0-2|=2$, $|2-4|=2$, $|4-1|=3$, $|1-3|=2$. Distance 2 checks: $|0-4|=4$, $|2-1|=1$, $|4-3|=1$. *All constraints hold, so ℓ_V is a valid SuperHyperGraph $L(2,1)$ labeling. In this case we exactly recover the classical graph labeling.*

Example 2.10 (An $n=1$ SuperHyperGraph with an $L(2,1)$–type labeling on overlapping sets). *Let $V_0=\{a,b,c,d\}$ and consider the 1–level supervertices*

$$A=\{a,b\},\quad B=\{b,c\},\quad C=\{c,d\},\quad D=\{a,d\}\ \in\ \mathscr{P}(V_0).$$

Set $V=\{A,B,C,D\}$ and define superedges

$$E_1=\{A,B,C\},\qquad E_2=\{A,D\},\qquad E_3=\{C,D\}.$$

Thus $\mathsf{SHG}^{(1)}=(V,\{E_1,E_2,E_3\})$. Its 2–section $G(\mathsf{SHG}^{(1)})$ has edges

$$\{A,B\},\ \{A,C\},\ \{B,C\}\ \ (\textit{from } E_1),\qquad \{A,D\}\ \ (\textit{from } E_2),\qquad \{C,D\}\ \ (\textit{from } E_3).$$

Let $L_V=\mathbb{Z}$ and impose the $L(2,1)$ schema from Remark 2.8 with distances taken in $G(\mathsf{SHG}^{(1)})$. Define the labeling

$$\ell_V(A)=0,\qquad \ell_V(B)=2,\qquad \ell_V(C)=5,\qquad \ell_V(D)=7.$$

Adjacency (distance 1) checks:

$$|0-2|=2\ (\{A,B\}),\quad |0-5|=5\ (\{A,C\}),\quad |2-5|=3\ (\{B,C\}),\quad |0-7|=7\ (\{A,D\}),\quad |5-7|=2\ (\{C,D\}).$$

Distance 2 check: *B and D have distance 2 (via A or C), and $|2-7|=5\geq 1$. Hence ℓ_V satisfies the $L(2,1)$ constraints on this genuinely superhyper (level $n=1$) instance. Note that the vertices here are* sets of base elements*, and superedges may have size 3, a setting that goes beyond ordinary graphs.*

Theorem 2.11 (SuperHyperGraph labeling generalizes graph and hypergraph labeling). *Let (G,Ψ) be any graph–labeling problem on a simple graph $G=(V,E_G)$, with predicate $\Psi(G,\lambda)$ expressed in terms of adjacency/distance in G and (in)equalities among labels. Let (H,Ψ) be any hypergraph–labeling problem on a hypergraph $H=(V,E_H)$ where distances are computed in the hypergraph 2–section. Then there exist $n\in\{0\}$, a SuperHyperGraph $\mathsf{SHG}^{(n)}$, and a labeling schema Φ such that:*

$$\exists\lambda:V\to L \text{ with } \Psi(G,\lambda)\iff \exists\ell_V:V\to L \text{ with } \Phi(\mathsf{SHG}^{(0)},\ell_V),$$

and

$$\exists\lambda:V\to L \text{ with } \Psi(H,\lambda)\iff \exists\ell_V:V\to L \text{ with } \Phi(\mathsf{SHG}^{(0)},\ell_V).$$

Consequently, SuperHyperGraph labeling strictly contains graph labeling (the case of 2–uniform hyperedges) and hypergraph labeling (the case $n=0$ with general hyperedges).

Proof. For the graph case, let $n=0$, $V_0:=V$, and define $\mathsf{SHG}^{(0)}=(V,E)$ with $E:=\{\{u,v\}\subseteq V\mid uv\in E_G\}$. Then $G\big(\mathsf{SHG}^{(0)}\big)=G$ by construction. Define $\Phi\big(\mathsf{SHG}^{(0)},\ell_V\big):=\Psi\big(G(\mathsf{SHG}^{(0)}),\ell_V\big)$. Hence $\Psi(G,\lambda)$ holds iff $\Phi(\mathsf{SHG}^{(0)},\lambda)$ holds, giving the first equivalence.
For the hypergraph case, again take $n=0$, $V_0:=V$, and set $\mathsf{SHG}^{(0)}=(V,E_H)$. By definition of the hypergraph 2–section, $G\big(\mathsf{SHG}^{(0)}\big)$ is exactly the primal graph used to measure distances in $\Psi(H,\cdot)$. Put $\Phi\big(\mathsf{SHG}^{(0)},\ell_V\big):=\Psi\big(H,\ell_V\big)$, interpreting all distance/adjacency relations through $G\big(\mathsf{SHG}^{(0)}\big)$. The same identity–of–structures argument yields the second equivalence.
Strict containment follows since for $n\geq 1$ one can add constraints that speak about the internal structure of n–level supervertices (e.g. intersection/nonintersection of members when $V\subseteq\mathscr{P}(V_0)$), which cannot be expressed on ordinary graphs nor on hypergraphs with $n=0$ without expanding the vertex set. □

2.3. Graph MultiLabeling

Graph MultiLabeling assigns multiple simultaneous labels to vertices and edges, supporting layered constraints, diverse applications, and richer graph optimization models.

Definition 2.12 (Graph MultiLabeling). *Fix nonnegative integers p,q. For a graph $G=(V,E)$, choose nonempty* vertex-label alphabets $L_V^{(1)},\dots,L_V^{(p)}$ *and nonempty* edge-label alphabets $L_E^{(1)},\dots,L_E^{(q)}$. *A* Graph MultiLabeling *on G is the tuple of maps*

$$\ell_V=\big(\ell_V^{(1)},\dots,\ell_V^{(p)}\big),\qquad \ell_V^{(a)}:V\to L_V^{(a)}\ (1\leq a\leq p),$$

$$\ell_E=\big(\ell_E^{(1)},\dots,\ell_E^{(q)}\big),\qquad \ell_E^{(b)}:E\to L_E^{(b)}\ (1\leq b\leq q).$$

Equivalently, $\ell_V:V\to\prod_{a=1}^{p}L_V^{(a)}$, $\ell_E:E\to\prod_{b=1}^{q}L_E^{(b)}$ with $\ell_V(v)=\big(\ell_V^{(1)}(v),\dots,\ell_V^{(p)}(v)\big)$ and similarly for edges.
A MultiLabeling schema *is a first-order predicate*

$$\Phi\big(G;\ell_V,\ell_E\big)$$

built from adjacency/distances in G, the incidence relation $u\in e$, the label components $\ell_V^{(a)}(u)$, $\ell_E^{(b)}(e)$, and fixed relations on these alphabets (e.g. equality, order, arithmetic, or application-specific constraints). We say that (ℓ_V,ℓ_E) is a valid Graph MultiLabeling for Φ *if $\Phi\big(G;\ell_V,\ell_E\big)$ holds.*

Remark 2.13 (Typical coordinatewise constraints)**.** *Many familiar labeling families appear as single coordinates:*

- *Proper coloring on coordinate a:* $L_V^{(a)} = \{1,\dots,k\}$ *and* $(\forall\{u,v\}\in E)\ \ell_V^{(a)}(u) \neq \ell_V^{(a)}(v)$.
- $L(p,q)$ *on coordinate a:* $L_V^{(a)} \subseteq \mathbb{Z}$ *with*

$$\mathrm{dist}_G(u,v) = 1 \Rightarrow |\ell_V^{(a)}(u) - \ell_V^{(a)}(v)| \geq p, \quad \mathrm{dist}_G(u,v) = 2 \Rightarrow |\ell_V^{(a)}(u) - \ell_V^{(a)}(v)| \geq q.$$

- *Edge capacities on coordinate b:* $L_E^{(b)} = \{1,\dots,C\}$ *with cross-constraints such as* $\ell_E^{(b)}(\{u,v\}) \geq f\big(\ell_V^{(a)}(u), \ell_V^{(a)}(v)\big)$ *for a fixed function* f.

Coordinates may also be coupled, *e.g. requiring that a time-slot label and a color label jointly avoid conflicts.*

Example 2.14 (Two-coordinate vertex MultiLabeling on a path: coloring + $L(2,1)$)**.** *Let* $G = P_5$ *with vertices* $v_1,\dots,v_5$ *and edges* v_iv_{i+1} $(i = 1,\dots,4)$*. Choose* $p = 2$*,* $q = 0$ *with*

$$L_V^{(1)} = \{r,g,b\} \quad \textit{(colors)}, \qquad L_V^{(2)} = \mathbb{Z} \quad \textit{(integers)}.$$

Define $\ell_V = (\ell_V^{(1)}, \ell_V^{(2)})$ *by*

$$\ell_V^{(1)} :\ (v_1,\dots,v_5) \mapsto (r,\ g,\ b,\ g,\ r),$$

$$\ell_V^{(2)} :\ (v_1,\dots,v_5) \mapsto (0,\ 2,\ 4,\ 1,\ 3).$$

Schema Φ *requires simultaneously:*

(C) Proper coloring *on coordinate* 1*: for each edge* v_iv_{i+1}*,* $\ell_V^{(1)}(v_i) \neq \ell_V^{(1)}(v_{i+1})$.
(N) $L(2,1)$ *on coordinate* 2*: for all distinct* u,v*,*

$$\mathrm{dist}_G(u,v) = 1 \Rightarrow |\ell_V^{(2)}(u) - \ell_V^{(2)}(v)| \geq 2, \qquad \mathrm{dist}_G(u,v) = 2 \Rightarrow |\ell_V^{(2)}(u) - \ell_V^{(2)}(v)| \geq 1.$$

Verification. *(C) Adjacent pairs are* (r,g)*,* (g,b)*,* (b,g)*,* (g,r)*, all unequal. (N) Adjacent gaps:* $|0-2| = 2$*,* $|2-4| = 2$*,* $|4-1| = 3$*,* $|1-3| = 2$*. Distance-*2 *gaps:* $|0-4| = 4$*,* $|2-1| = 1$*,* $|4-3| = 1$*. Hence* $(\ell_V, \ell_E = \varnothing)$ *satisfies* Φ*. This is a genuine* multi-*labeling: two coordinated vertex labelings enforced at once.*

Example 2.15 (Vertex & edge MultiLabeling with cross-constraints)**.** *Let G be the* 4*-cycle* C_4 *on* v_1,v_2,v_3,v_4 *(in order). Take* $p = 1$*,* $q = 1$ *with* $L_V^{(1)} = \{A,B\}$ *(two roles) and* $L_E^{(1)} = \{1,2\}$ *(capacity classes). Define*

$$\ell_V^{(1)}(v_1) = A,\ \ell_V^{(1)}(v_2) = B,\ \ell_V^{(1)}(v_3) = A,\ \ell_V^{(1)}(v_4) = B,$$

$$\ell_E^{(1)}(\{v_1,v_2\}) = 2,\ \ell_E^{(1)}(\{v_2,v_3\}) = 1,\ \ell_E^{(1)}(\{v_3,v_4\}) = 2,\ \ell_E^{(1)}(\{v_4,v_1\}) = 1.$$

Schema Φ *requires:*

- *Adjacent vertices must have different roles:* $\ell_V^{(1)}(u) \neq \ell_V^{(1)}(v)$ *for all* $\{u,v\} \in E$ *(a* 2*-coloring).*
- *Edge capacity must dominate the* role disparity*:*

$$\ell_E^{(1)}(\{u,v\}) \ \geq\ \begin{cases} 2, & \textit{if } \{\ell_V^{(1)}(u), \ell_V^{(1)}(v)\} = \{A,B\}, \\ 1, & \textit{if } \ell_V^{(1)}(u) = \ell_V^{(1)}(v). \end{cases}$$

Verification. *The vertex roles alternate* A,B,A,B*, so the first condition holds. Each edge has endpoints of different roles, and is labeled capacity* 2 *or* 1 *as above; precisely those with alternating endpoints have capacity* 2*, satisfying the inequality. Thus* (ℓ_V, ℓ_E) *is a valid MultiLabeling with coupled vertex/edge constraints.*

Theorem 2.16 (Graph MultiLabeling generalizes classical graph labeling)**.** *Let* (G,Ψ) *be any classical graph-labeling problem on* $G = (V,E)$*: i.e., choose a single label set L and a predicate* $\Psi(G,\lambda)$ *over mappings* $\lambda : V \to L$ *(or* $\lambda : E \to L$*) that is expressed using graph structure and relations on L. Then there exists a MultiLabeling schema* Φ *with* $p = 1$ *(and* $q = 0$ *for a vertex-labeling, or* $q = 1$*,* $p = 0$ *for an edge-labeling) such that*

$$\exists\lambda \textit{ with } \Psi(G,\lambda) \quad \Longleftrightarrow \quad \exists(\ell_V,\ell_E) \textit{ with } \Phi\big(G;\ell_V,\ell_E\big).$$

Proof. Vertex-labeling case. Take $p = 1$, $q = 0$, and set $L_V^{(1)} := L$. Define $\Phi\big(G;(\ell_V^{(1)}),\varnothing\big) := \Psi\big(G,\ell_V^{(1)}\big)$. Then λ satisfies Ψ iff $\ell_V^{(1)} := \lambda$ satisfies Φ; existence is equivalent.

Edge-labeling case is identical with $p = 0$, $q = 1$, $L_E^{(1)} := L$, and $\Phi\big(G;\varnothing,(\ell_E^{(1)})\big) := \Psi\big(G,\ell_E^{(1)}\big)$. □

2.4. HyperGraph MultiLabeling

HyperGraph MultiLabeling provides multiple coordinated labels to vertices and hyperedges, generalizing graph multilabeling and enabling multi-role representations of complex relationships.

Definition 2.17 (Primal (2–section) of a hypergraph)**.** *Given $H=(V,E)$, its* primal graph *(also 2–section) is*

$$G(H) := (V,E') \qquad \text{with} \qquad E' := \big\{\{u,v\} \subseteq V \mid \exists e \in E :\ \{u,v\} \subseteq e\big\}.$$

We write $\mathrm{dist}_H(u,v)$ *for the usual shortest–path distance of $u,v \in V$ taken in $G(H)$.*

Definition 2.18 (HyperGraph MultiLabeling)**.** *Fix nonnegative integers p,q. Let $H=(V,E)$ be a hypergraph. Choose nonempty* vertex label alphabets $L_V^{(1)},\dots,L_V^{(p)}$ *and nonempty* hyperedge label alphabets $L_E^{(1)},\dots,L_E^{(q)}$*. A* HyperGraph MultiLabeling *on H consists of the coordinate maps*

$$\ell_V = \big(\ell_V^{(1)},\dots,\ell_V^{(p)}\big), \quad \ell_V^{(a)} : V \to L_V^{(a)}\ (1 \le a \le p),$$

$$\ell_E = \big(\ell_E^{(1)},\dots,\ell_E^{(q)}\big), \quad \ell_E^{(b)} : E \to L_E^{(b)}\ (1 \le b \le q).$$

Equivalently, a single vertex map $\ell_V : V \to \prod_{a=1}^{p} L_V^{(a)}$ together with a single edge map $\ell_E : E \to \prod_{b=1}^{q} L_E^{(b)}$.
A MultiLabeling schema *is a first–order predicate*

$$\Phi\big(H;\ell_V,\ell_E\big)$$

built from the incidence relation $v \in e$, the distance dist_H *on V, (and possibly fixed relations/operations on the alphabets, such as $=$, $\neq$, order, arithmetic, etc.). We say (ℓ_V,ℓ_E) is a* valid HyperGraph MultiLabeling (for Φ) *if $\Phi\big(H;\ell_V,\ell_E\big)$ holds.*

Remark 2.19 (Typical coordinatewise and cross–coordinate constraints)**.** *The schema Φ can encode, e.g.:*

- ***Strong hyperedge coloring** on a vertex coordinate a: for every $e \in E$, the set $\{\ell_V^{(a)}(v) : v \in e\}$ is pairwise distinct.*
- *$L(p,q)$**–type spacing** on a vertex coordinate a: if $\mathrm{dist}_H(u,v)=1$ then $|\ell_V^{(a)}(u)-\ell_V^{(a)}(v)| \ge p$, and if $\mathrm{dist}_H(u,v)=2$ then $|\ell_V^{(a)}(u)-\ell_V^{(a)}(v)| \ge q$.*
- ***Vertex–edge coupling**: for each $e \in E$, a constraint linking $\ell_E^{(b)}(e)$ to an aggregate of $\{\ell_V^{(a)}(v) : v \in e\}$ (sum, max, cardinality, etc.).*

Example 2.20 (Two–coordinate vertex & one–coordinate edge MultiLabeling on a 3–uniform hypergraph)**.** *Let $V=\{a,b,c,d\}$ and $E=\{e_1,e_2\}$ with $e_1=\{a,b,c\}$ and $e_2=\{b,c,d\}$. Choose $p=2$, $q=1$ with*

$$L_V^{(1)} = \{R,G,B\} \quad (colors), \qquad L_V^{(2)} = \mathbb{Z}_{\ge 0} \quad (workload),$$

$$L_E^{(1)} = \mathbb{Z}_{\ge 0} \quad (edge\ deadline).$$

Define the labeling:

$$\ell_V^{(1)}(a)=R,\ \ell_V^{(1)}(b)=G,\ \ell_V^{(1)}(c)=B,\ \ell_V^{(1)}(d)=G,$$

$$\ell_V^{(2)}(a)=1,\ \ell_V^{(2)}(b)=2,\ \ell_V^{(2)}(c)=1,\ \ell_V^{(2)}(d)=3,$$

$$\ell_E^{(1)}(e_1)=5, \qquad \ell_E^{(1)}(e_2)=7.$$

Schema Φ requires simultaneously:

(S) Strong hyperedge coloring *on coordinate 1: within each $e \in E$, the colors are pairwise distinct.*
(C) Capacity coupling*: for each $e \in E$,*

$$\sum_{v \in e} \ell_V^{(2)}(v) \ \le\ \ell_E^{(1)}(e).$$

Verification. *(S) In $e_1=\{a,b,c\}$ we have (R,G,B) distinct; in $e_2=\{b,c,d\}$ we have (G,B,G)* not *all distinct. Thus the strong constraint would fail on e_2. Instead, choose a* weak *variant for e_2: "at least two colors in each hyperedge"; then e_2 uses $\{G,B\}$ and passes. (C) e_1: $1+2+1=4\le 5$; e_2: $2+1+3=6\le 7$. Hence with weak hyperedge coloring the tuple (ℓ_V,ℓ_E) satisfies Φ.*

Example 2.21 (Distance–aware MultiLabeling (an $L(2,1)$–type coordinate) plus edge aggregation)**.** *Let $H=(V,E)$ with $V=\{x_1,x_2,x_3,x_4\}$ and*

$$E = \big\{\{x_1,x_2,x_3\},\ \{x_2,x_3,x_4\}\big\}.$$

Its primal graph $G(H)$ has edges $\{x_1,x_2\}$, $\{x_1,x_3\}$, $\{x_2,x_3\}$, $\{x_2,x_4\}$, $\{x_3,x_4\}$. Choose $p=2$, $q=1$ with

$$L_V^{(1)} = \mathbb{Z} \quad (frequency\ channel), \qquad L_V^{(2)} = \{0,1\} \quad (role), \qquad L_E^{(1)} = \mathbb{Z}_{\ge 0} \quad (edge\ cost).$$

Define

$$\ell_V^{(1)}(x_1)=0,\ \ell_V^{(1)}(x_2)=2,\ \ell_V^{(1)}(x_3)=5,\ \ell_V^{(1)}(x_4)=7,$$

$$\ell_V^{(2)}(x_1)=0,\ \ell_V^{(2)}(x_2)=1,\ \ell_V^{(2)}(x_3)=0,\ \ell_V^{(2)}(x_4)=1,$$

$$\ell_E^{(1)}(\{x_1,x_2,x_3\})=5, \qquad \ell_E^{(1)}(\{x_2,x_3,x_4\})=7.$$

Schema Φ requires:

$(L(2,1))$ *On coordinate* 1 *(channels): for distinct* $u,v\in V$,

$$\mathrm{dist}_H(u,v)=1\Rightarrow|\ell_V^{(1)}(u)-\ell_V^{(1)}(v)|\geq 2,\quad \mathrm{dist}_H(u,v)=2\Rightarrow|\ell_V^{(1)}(u)-\ell_V^{(1)}(v)|\geq 1.$$

(R) *On coordinate* 2 *(roles): each hyperedge contains both roles* 0 *and* 1.

(A) Edge aggregation*: for each* $e\in E$,

$$\ell_E^{(1)}(e)=\max\{\ell_V^{(1)}(v):v\in e\}.$$

Distances are taken in $G(H)$*, which is the* K_4 *minus edges* $\{x_1,x_4\}$*; hence* $\mathrm{dist}_H(x_1,x_4)=2$*. All adjacent pairs satisfy channel gaps:* $|0-2|=2$, $|0-5|=5$, $|2-5|=3$, $|2-7|=5$, $|5-7|=2$*. Distance 2 pair* (x_1,x_4) *has* $|0-7|=7\geq 1$*. (R) Each hyperedge* $\{x_1,x_2,x_3\}$ *and* $\{x_2,x_3,x_4\}$ *contains roles* $\{0,1\}$*. (A) For* $\{x_1,x_2,x_3\}$ *the max channel is* $\max\{0,2,5\}=5$*, matching* $\ell_E^{(1)}=5$*; for* $\{x_2,x_3,x_4\}$ *the max is* $\max\{2,5,7\}=7$*, matching* $\ell_E^{(1)}=7$*. Thus* (ℓ_V,ℓ_E) *satisfies* Φ.

Theorem 2.22 (HyperGraph MultiLabeling generalizes hypergraph labeling and graph MultiLabeling)**.** *Let* (H,Ψ) *be any (single–coordinate) hypergraph labeling problem, with* $\Psi(H,\lambda)$ *a predicate over* $\lambda:V\to L$ *(or* $\lambda:E\to L$*) using only the hypergraph structure (incidence and/or* dist_H*) and relations on L. Let* (G,Φ_{GML}) *be any Graph MultiLabeling instance on a simple graph* $G=(V,E_G)$ *with p vertex and q edge coordinates.*
Then:

1. ***(Hypergraph labeling*** $\subseteq$ ***HyperGraph MultiLabeling)*** *There exists a schema* Φ *with either* $(p,q)=(1,0)$ *(vertex case) or* $(p,q)=(0,1)$ *(edge case) such that*

$$\exists\lambda \text{ with } \Psi(H,\lambda)\iff\exists(\ell_V,\ell_E)\text{ with }\Phi\big(H;\ell_V,\ell_E\big),$$

where $\ell_V^{(1)}=\lambda$ *(or* $\ell_E^{(1)}=\lambda$*).*

2. ***(Graph MultiLabeling*** $\subseteq$ ***HyperGraph MultiLabeling)*** *There exists a hypergraph* $\widehat{H}=(V,\widehat{E})$ *with* $\widehat{E}:=\{\{u,v\}\subseteq V:uv\in E_G\}$ *and a schema* $\widehat{\Phi}$ *such that*

$$\exists(\ell_V,\ell_E)\text{ with }\Phi_{\mathrm{GML}}\big(G;\ell_V,\ell_E\big)\iff\exists(\ell_V,\ell_E)\text{ with }\widehat{\Phi}\big(\widehat{H};\ell_V,\ell_E\big).$$

Hence HyperGraph MultiLabeling strictly contains both classical hypergraph labeling and graph MultiLabeling.

Proof. (1) Vertex case. Take $p=1,q=0$, set $L_V^{(1)}:=L$, and define $\Phi\big(H;(\ell_V^{(1)}),\varnothing\big):=\Psi\big(H,\ell_V^{(1)}\big)$. Then $\Psi(H,\lambda)$ holds iff $\Phi\big(H;(\lambda),\varnothing\big)$ holds; existence is equivalent. Edge case is identical with $p=0,q=1$ and $L_E^{(1)}:=L$.
(2) Let $\widehat{H}=(V,\widehat{E})$ be the 2–uniform hypergraph obtained from G by setting one hyperedge for each graph edge. Then $G(\widehat{H})=G$, hence $\mathrm{dist}_{\widehat{H}}=\mathrm{dist}_G$. Define $\widehat{\Phi}(\widehat{H};\ell_V,\ell_E):=\Phi_{\mathrm{GML}}\big(G(\widehat{H});\ell_V,\ell_E\big)$. Any constraint in Φ_{GML} that refers to adjacency/distances or incidence in G is identically interpreted in $\widehat{H}$ via $G(\widehat{H})=G$, so the two existence statements are equivalent. ☐

2.5. SuperHyperGraph MultiLabeling

SuperHyperGraph MultiLabeling assigns multiple labels to supervertices and superedges, generalizing both hypergraph multilabeling and superhypergraph labeling for hierarchical multi-dimensional applications.

Definition 2.23 (Primal (2–section) of a SuperHyperGraph and distance)**.** *Let* $\mathrm{SHG}^{(n)}=(V,E)$ *be an n–SuperHyperGraph, i.e.* $V\subseteq\mathscr{P}^n(V_0)$ *and* $\varnothing\neq E\subseteq\mathscr{P}(V)\setminus\{\varnothing\}$*. Its* primal graph *(or* 2*–section) is*

$$G\Big(\mathrm{SHG}^{(n)}\Big):=(V,E')\quad\text{with}\quad E':=\big\{\{u,v\}\subseteq V\ \big|\ \exists F\in E:\ \{u,v\}\subseteq F\big\}.$$

The vertex distance $\mathrm{dist}_{\mathrm{SHG}}(x,y)$ *is the usual shortest–path distance between* $x,y\in V$ *computed in* $G\Big(\mathrm{SHG}^{(n)}\Big)$.

Definition 2.24 (Base support (flattening))**.** *For* $n\geq 0$ *define recursively a map* $\mathrm{flat}_n:\mathscr{P}^n(V_0)\to\mathscr{P}(V_0)$ *by*

$$\mathrm{flat}_0(v):=\{v\}\ \ (v\in V_0),\qquad \mathrm{flat}_{n+1}(X):=\bigcup_{Y\in X}\mathrm{flat}_n(Y)\ \ (X\in\mathscr{P}^{n+1}(V_0)).$$

Thus, for a supervertex $X\in V\subseteq\mathscr{P}^n(V_0)$*, the set* $\mathrm{flat}_n(X)\subseteq V_0$ *collects all base elements of* V_0 *that occur anywhere inside* X.

Definition 2.25 (SuperHyperGraph MultiLabeling)**.** *Fix integers* $p,q\geq 0$ *and an n–SuperHyperGraph* $\mathrm{SHG}^{(n)}=(V,E)$*. Choose nonempty* vertex label alphabets $L_V^{(1)},\dots,L_V^{(p)}$ *and nonempty* edge label alphabets $L_E^{(1)},\dots,L_E^{(q)}$*. A* SuperHyperGraph MultiLabeling *consists of*

$$\ell_V=\big(\ell_V^{(1)},\dots,\ell_V^{(p)}\big),\quad \ell_V^{(a)}:V\to L_V^{(a)}\quad(1\leq a\leq p),$$

$$\ell_E=\big(\ell_E^{(1)},\dots,\ell_E^{(q)}\big),\quad \ell_E^{(b)}:E\to L_E^{(b)}\quad(1\leq b\leq q).$$

Equivalently, a single vertex map $\ell_V:V\to\prod_{a=1}^{p}L_V^{(a)}$ *and a single edge map* $\ell_E:E\to\prod_{b=1}^{q}L_E^{(b)}$*. A* schema *is a first–order predicate*

$$\Phi\Big(\mathrm{SHG}^{(n)};\ell_V,\ell_E\Big)$$

built from the incidence relation $v\in F$ *(*$v\in V$*,* $F\in E$*), the distance* $\mathrm{dist}_{\mathrm{SHG}}$*, the operators* flat_n*, basic set–theoretic/cardinality operations on* $\mathscr{P}(V_0)$*, and given relations/operations on the alphabets. The pair* (ℓ_V,ℓ_E) *is a* valid SuperHyperGraph MultiLabeling (for Φ) *if* $\Phi\Big(\mathrm{SHG}^{(n)};\ell_V,\ell_E\Big)$ *holds.*

Remark 2.26 (Typical constraints that Φ may express).

- ***Distance–aware separation on a vertex coordinate*** a*: if* $\mathrm{dist}_{\mathrm{SHG}}(x,y)=1$ *then* $|\ell_V^{(a)}(x)-\ell_V^{(a)}(y)|\geq\lambda_1$*, and if* $\mathrm{dist}_{\mathrm{SHG}}(x,y)=2$ *then* $|\ell_V^{(a)}(x)-\ell_V^{(a)}(y)|\geq\lambda_2$.
- ***Support cardinality on a vertex coordinate*** a*:* $\ell_V^{(a)}(X)=\big|\mathrm{flat}_n(X)\big|$ *for all* $X\in V$.
- ***Superedge aggregation on an edge coordinate*** b*:* $\ell_E^{(b)}(F)=g\big(\{\mathrm{flat}_n(X):X\in F\}\big)$ *for a fixed aggregator* g *(e.g. union size, intersection size, Jaccard index, maximum of a vertex coordinate, etc.).*

Example 2.27 (A $(p,q)=(2,1)$ multilabel on a level-$n=1$ SuperHyperGraph)**.** *Let* $V_0=\{a,b,c,d\}$*. Consider* $n=1$ *so supervertices are subsets of* V_0*. Let*

$$V=\{X_1,X_2,X_3\}:=\{\{a,b\},\ \{b,c\},\ \{c,d\}\},\qquad E=\{F_1,F_2\}:=\{\{X_1,X_2\},\ \{X_2,X_3\}\}.$$

Hence $G\Big(\mathrm{SHG}^{(1)}\Big)$ *is the path* X_1–X_2–X_3*. Choose alphabets*

$$L_V^{(1)}=\{R,G,B\}\quad(colors),\qquad L_V^{(2)}=\mathbb{N}\quad(support\ size),\qquad L_E^{(1)}=\mathbb{N}\quad(coverage\ size).$$

Define the labels

$$\ell_V^{(1)}(X_1)=R,\ \ell_V^{(1)}(X_2)=G,\ \ell_V^{(1)}(X_3)=B,\quad \ell_V^{(2)}(X_i)=\big|\mathrm{flat}_1(X_i)\big|=|X_i|=2\ \ (i=1,2,3),$$

$$\ell_E^{(1)}(F_1)=\big|\mathrm{flat}_1(X_1)\cup\mathrm{flat}_1(X_2)\big|=\big|\{a,b\}\cup\{b,c\}\big|=3,$$

$$\ell_E^{(1)}(F_2)=\big|\mathrm{flat}_1(X_2)\cup\mathrm{flat}_1(X_3)\big|=\big|\{b,c\}\cup\{c,d\}\big|=3.$$

Schema Φ *imposes simultaneously:*

(Col) *Adjacent supervertices receive different colors on coordinate* 1.
(Sup) *For all* $X\in V$, $\ell_V^{(2)}(X)=\big|\mathrm{flat}_1(X)\big|$.
(Agg) *For all* $F\in E$, $\ell_E^{(1)}(F)=\big|\bigcup_{X\in F}\mathrm{flat}_1(X)\big|$.

The given $\big(\ell_V,\ell_E\big)$ *satisfies (Col), (Sup), and (Agg), hence it is a valid SuperHyperGraph MultiLabeling.*

Example 2.28 (A distance–aware multilabel on a level-$n=2$ SuperHyperGraph)**.** *Let* $V_0=\{1,2,3\}$ *and* $n=2$*. Define supervertices*

$$A:=\{\{1\},\{1,2\}\},\qquad B:=\{\{2\},\{2,3\}\}\ \in\ \mathscr{P}^2(V_0),$$

and set $V=\{A,B\}$, $E=\{\{A,B\}\}$*. Then*

$$\mathrm{flat}_2(A)=\{1,2\},\qquad \mathrm{flat}_2(B)=\{2,3\}.$$

Choose alphabets $L_V^{(1)}=\mathbb{Z}$ *(channels),* $L_V^{(2)}=\mathbb{N}$ *(support size),* $L_E^{(1)}=\mathbb{N}$ *(overlap size). Define*

$$\ell_V^{(1)}(A)=0,\ \ \ell_V^{(1)}(B)=3\quad(gap\ 3),\qquad \ell_V^{(2)}(A)=2,\ \ \ell_V^{(2)}(B)=2,$$

$$\ell_E^{(1)}\big(\{A,B\}\big)=\big|\mathrm{flat}_2(A)\cap\mathrm{flat}_2(B)\big|=\big|\{1,2\}\cap\{2,3\}\big|=1.$$

Let Φ *require:*

$L(2,1)$ *On coordinate* 1*, if* $\mathrm{dist}_{\mathrm{SHG}}(X,Y)=1$ *then* $|\ell_V^{(1)}(X)-\ell_V^{(1)}(Y)|\geq 2$ *(here* $|0-3|=3\geq 2$*).*
(Sup) $\ell_V^{(2)}(X)=\big|\mathrm{flat}_2(X)\big|$ *for all* $X\in V$ *(true:* 2 *and* 2*).*
(Int) $\ell_E^{(1)}(F)=\big|\bigcap_{X\in F}\mathrm{flat}_2(X)\big|$ *for all* $F\in E$ *(true:* 1*).*

Thus (ℓ_V,ℓ_E) *satisfies* Φ *and is a valid multilabel.*

Theorem 2.29 (SuperHyperGraph MultiLabeling generalizes SuperHyperGraph Labeling and HyperGraph MultiLabeling)**.** *The framework in Definition 2.25 strictly contains:*

(i) SuperHyperGraph Labeling *(single–coordinate labeling on supervertices and/or superedges);*
(ii) HyperGraph MultiLabeling *(multi–coordinate labeling on ordinary hypergraphs).*

Proof. (i) Let a SuperHyperGraph labeling be given as a single map $\lambda_V:V\to L$ (vertex case) or $\lambda_E:E\to L$ (edge case), together with a predicate Ψ that uses only incidence, the primal distance and allowed relations on L. Take $p=1$, $q=0$ (vertex case) with $L_V^{(1)}:=L$, and set

$$\Phi\Big(\mathrm{SHG}^{(n)};\ (\ell_V^{(1)}),\ \varnothing\Big)\ :=\ \Psi\Big(\mathrm{SHG}^{(n)};\ \ell_V^{(1)}\Big).$$

Then λ_V satisfies Ψ iff $(\ell_V^{(1)},\varnothing)$ with $\ell_V^{(1)}=\lambda_V$ satisfies Φ. The edge case is identical with $p=0$, $q=1$.
(ii) Let $H=(V,E)$ be a (finite) hypergraph and consider any HyperGraph MultiLabeling instance on H (with p vertex and q edge coordinates and a schema Φ_{HG} based on the hypergraph incidence/distance). Realize H as an $n=0$ SuperHyperGraph by taking $V_0:=V$ and $\mathrm{SHG}^{(0)}:=(V,E)$; then $G\Big(\mathrm{SHG}^{(0)}\Big)$ coincides with the primal of H, so the same adjacency/distance is available. Define

$$\Phi\Big(\mathrm{SHG}^{(0)};\ell_V,\ell_E\Big)\ :=\ \Phi_{\mathrm{HG}}(H;\ell_V,\ell_E).$$

Thus every feasible HyperGraph MultiLabeling on H is a feasible SuperHyperGraph MultiLabeling on $\mathrm{SHG}^{(0)}$, and conversely. Therefore the latter generalizes the former. □

3. Conclusion

In this paper, we defined and study the mathematical properties of *Graph Labeling*, *HyperGraph Labeling*, *SuperHyperGraph Labeling*, *Graph MultiLabeling*, *HyperGraph MultiLabeling*, and *SuperHyperGraph MultiLabeling*. We anticipate that future work may explore extensions employing *Fuzzy Sets*[42–44], *Intuitionistic Fuzzy Sets*[45–48], *Neutrosophic Sets*[24, 49–51], Picture Fuzzy Sets [52–55], Bipolar Neutrosophic Sets [56–58], HyperFuzzy Sets [15, 59–62], and *Plithogenic Sets*[63–66].

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